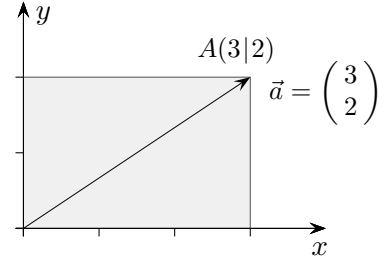


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↑ Vector Calculations Introduction

In physics, quantities such as force and velocity often occur which cannot be captured by a single number. They have a direction in addition to a certain amount and are therefore represented by arrows. For the beginning we limit ourselves to those arrows whose starting point lies in the coordinate origin. These arrows can be described by the coordinates of their end point $A(x|y)$. To distinguish them from the points the coordinates of the vectors are written below each other. Vectors are

either elements of \mathbb{R}^2 , for example $\vec{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ or of \mathbb{R}^3 , $\vec{b} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$.

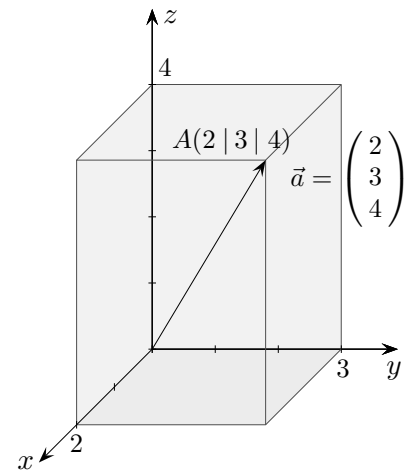


Here vectors are first used as position (location) vectors (this is theoretically sufficient, but later on it is completely impractical).

Vectors can be added.

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad \text{in space:} \quad \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ 5 \end{pmatrix}$$

The sum corresponds to the resultant of the parallelogram spanned by the the vectors to be added.

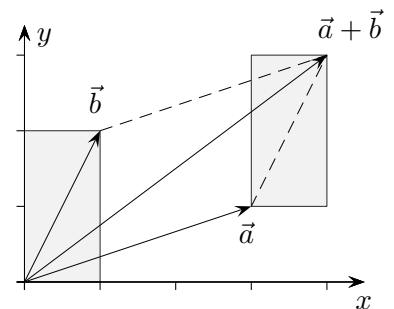


Vectors can be multiplied by a number (scalar):

$$3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$$

The multiplication with a positiv number only changes the length of the vector and not direction. In the multiplication with a negative scalar, the direction of the vector changes in the opposite direction.

Since numbers in vector calculations are called scalars for distinction this multiplication is called scalar multiplication (later we will get to know a scalar product). For scalars, besides r, s and t , the Greek letters λ lambda and μ my are used.

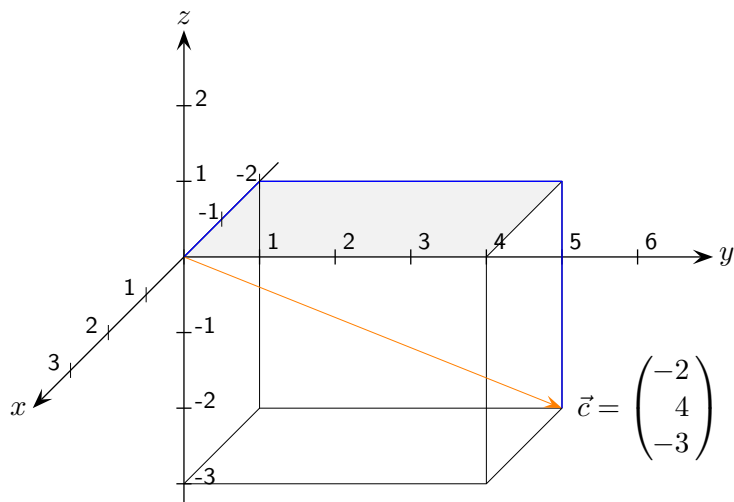
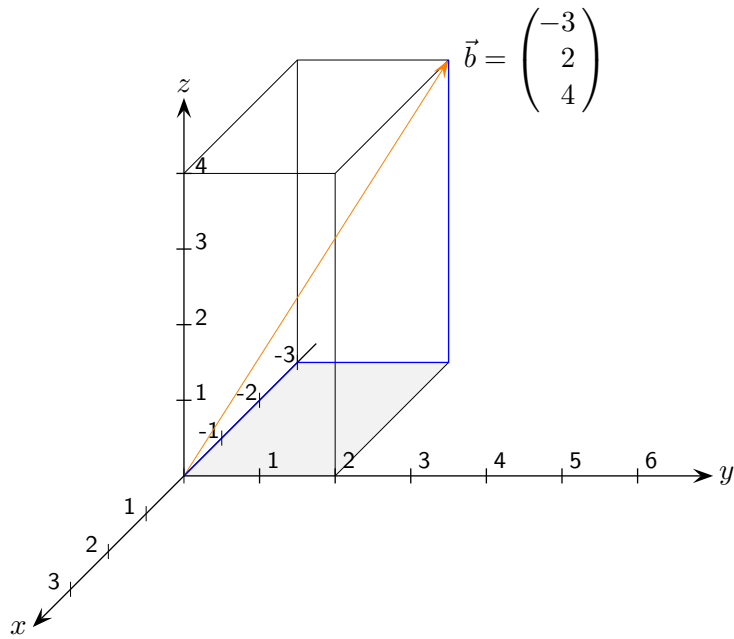
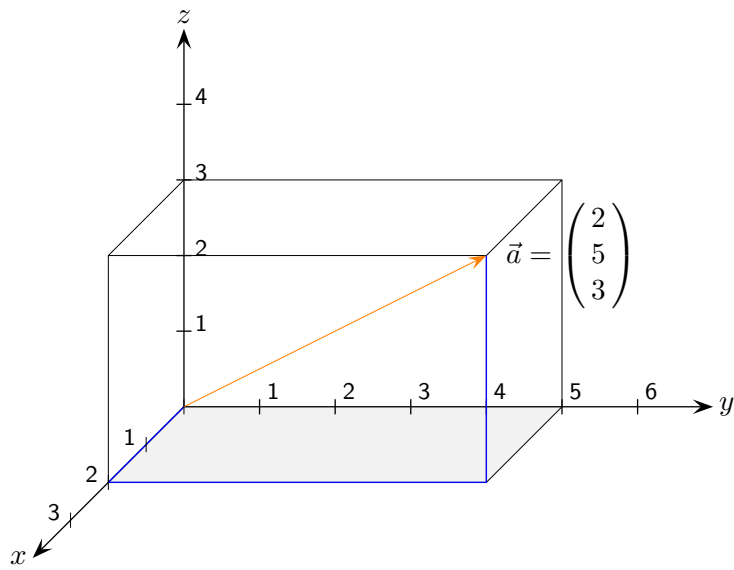


Exercise Draw the vectors:

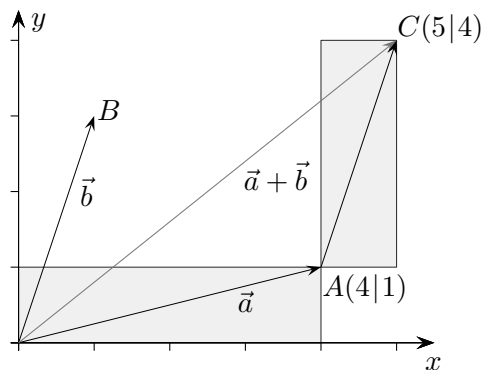
a) $\vec{a} = \begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}$ b) $\vec{b} = \begin{pmatrix} -3 \\ 2 \\ 4 \end{pmatrix}$ c) $\vec{c} = \begin{pmatrix} -2 \\ 4 \\ -3 \end{pmatrix}$

The vector calculus constitutes together with the differential and integral calculus the foundation of higher mathematics. There are applications in the natural and economic sciences. The foundations were created by Hamilton (1805-65) and Grassmann (1809-77). The physicist Gibbs (1839-1903) developed the present form (scalar and vector product).

\vec{a} can be obtained by **stepwise change** of the components of $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$: $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix}$



↑ Vector addition



$$\vec{a} + \vec{b} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4+1 \\ 1+3 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$$

The arrow attached to A shows that for the addition of one vector \vec{a} (position vector) can be started and the other vector indicates the coordinate changes to get from A to C (direction vector \vec{b}).

The addition of vectors reveals the second possible interpretation of a vector: it contains the coordinate changes of a summand. The associated arrows then indicate a direction and are no longer necessarily bounded to the origin.

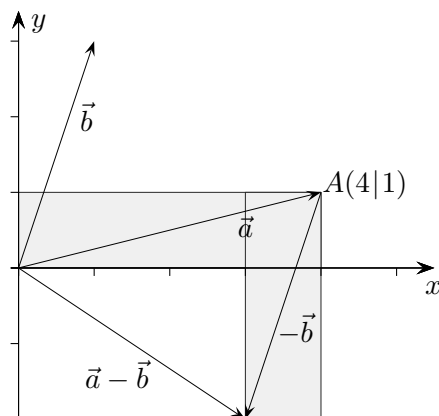
Which property (point or direction) is meant is obvious from the application. For vectors both interpretations are possible (simultaneously). Vectors in geometry are thus of hermaphroditic nature.

The aspect can be emphasized by the notation, $\vec{b} = \overrightarrow{OB}$ as position vector, $\vec{b} = \overrightarrow{AC}$ as direction vector.

$$\vec{a} - \vec{b} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4-1 \\ 1-3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

Subtraction of vectors is similar to addition

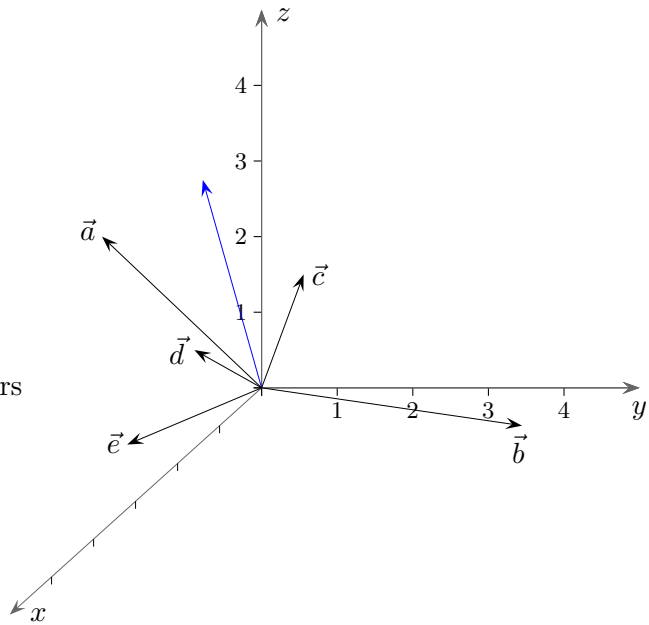
only the second vector will be mirrored to the opposite direction.



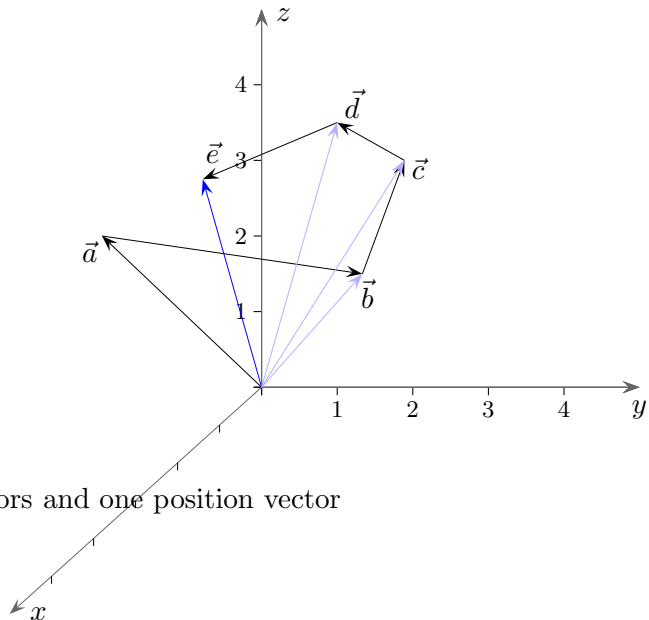
↑ Vector sum

$$\vec{a} + \vec{b} + \vec{c} + \vec{d} + \vec{e} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ -2 \\ -0,5 \end{pmatrix} + \begin{pmatrix} 0,5 \\ -1,5 \\ -0,5 \end{pmatrix} = \begin{pmatrix} 0,5 \\ -0,5 \\ 3 \end{pmatrix}$$

Visualization with position vectors

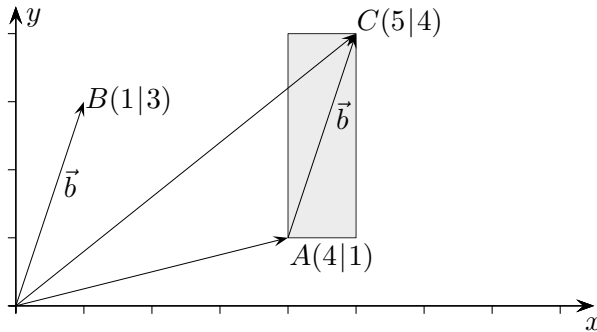


Visualization with direction vectors and one position vector



↑ Direction vector

Let a direction be determined by 2 points A and C .
What is the direction vector \vec{AC} ?



In simple cases, the direction vector (difference vector) can be read directly, $\vec{b} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

On the other hand, for $\vec{b} = \vec{AC}$: $\vec{AC} = \vec{OC} - \vec{OA}$ because $\vec{OA} + \vec{AC} = \vec{OC}$.

$$\vec{b} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \text{Note: „Tip minus tail“}$$

The top number of the vector tells you how many spaces or units to move in the positive x -direction and the bottom number is how many to move in the positive y -direction.

$$\vec{CA} = \vec{OA} - \vec{OC} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

↑ Visualized

How to visualize the vector for instance $\vec{a} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$?

The direction is given by:

Three steps in x -axis direction, two steps in y -axis direction.

The visualisation is suggested by the context of application.

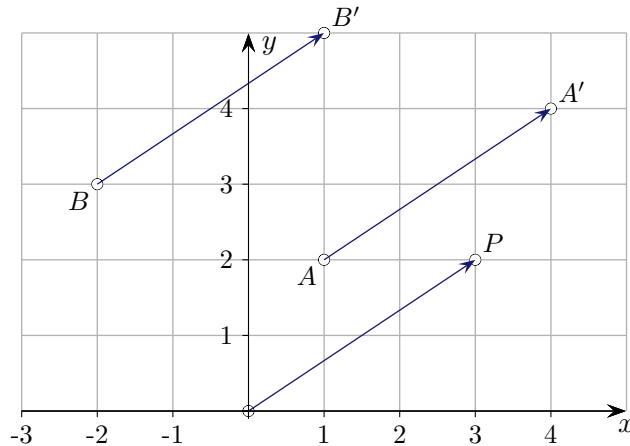
1) Position vector

We consider the vector \vec{a} as a position vector starting from the origin and defining a point $P(3 | 2)$.

2) Direction vector

Let us consider any point in the plane, for instance $A(1 | 2)$. By \vec{a} is defined a point A' whose coordinates we obtain by adding the vector components $A'(1 + 3 | 2 + 2)$, in short

$$\begin{aligned} \vec{OA} + \vec{a} &= \vec{OA'} \\ \vec{A} + \vec{a} &= \vec{A'} \end{aligned}$$



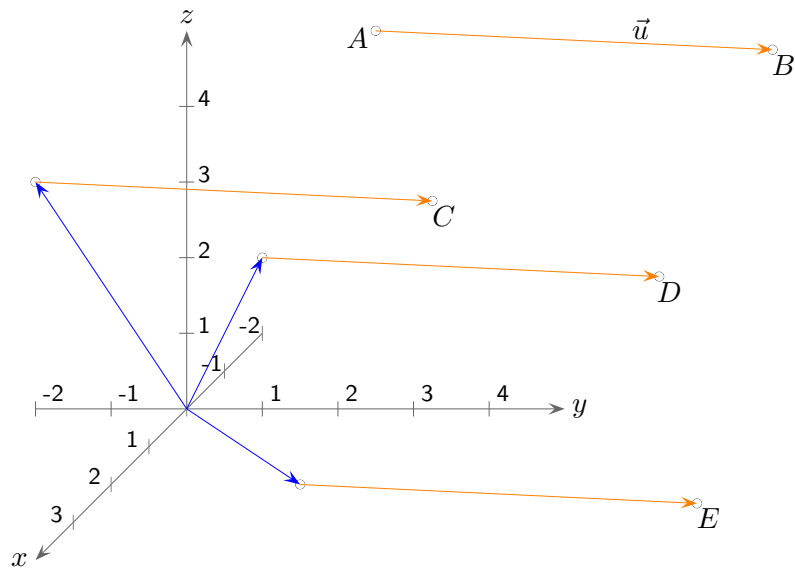
Let $B(-2 | 3)$ be given. By \vec{a} then the point $B'(-2 + 3 | 3 + 2)$ is determined.

$$\begin{aligned} \vec{OB} + \vec{a} &= \vec{OB'} \\ \vec{B} + \vec{a} &= \vec{B'} \end{aligned}$$

As a probe serves:

$$\begin{aligned} \vec{a} &= \vec{AA'} = \vec{OA'} - \vec{OA} \\ \vec{a} &= \vec{BB'} = \vec{OB'} - \vec{OB} \end{aligned}$$

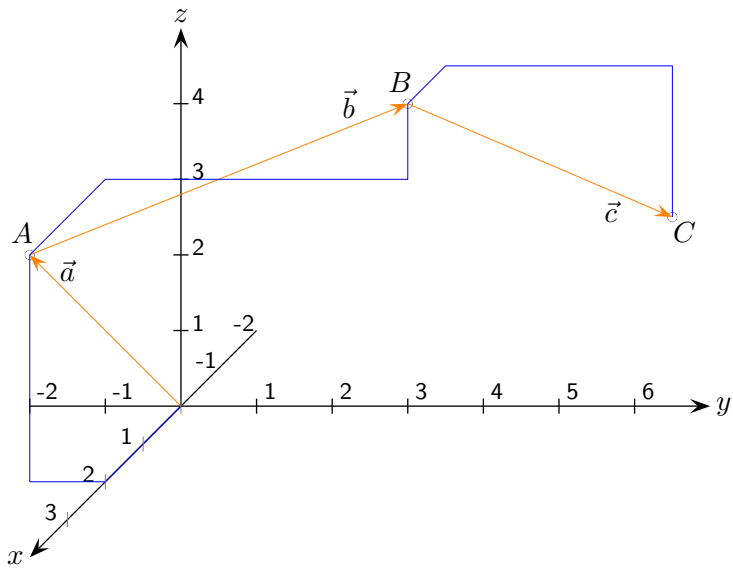
↑ Position vector/direction vector



By the points A and B a direction is given, with which each point can be moved in parallel. The direction vector $\vec{u} = \vec{AB} = \vec{OB} - \vec{OA}$ is attached here to the different position vectors (blue). This gives the position vectors of the points C , D and E .

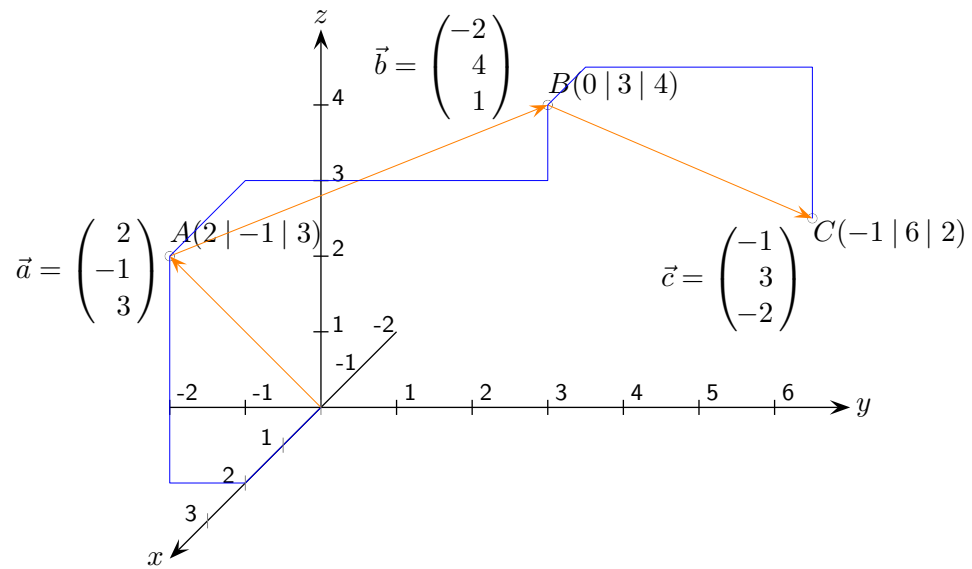
↑ Vectors and Points

Find the components of the vectors \vec{a} , \vec{b} and \vec{c} and the coordinates of the points A , B and C . All values are integers.

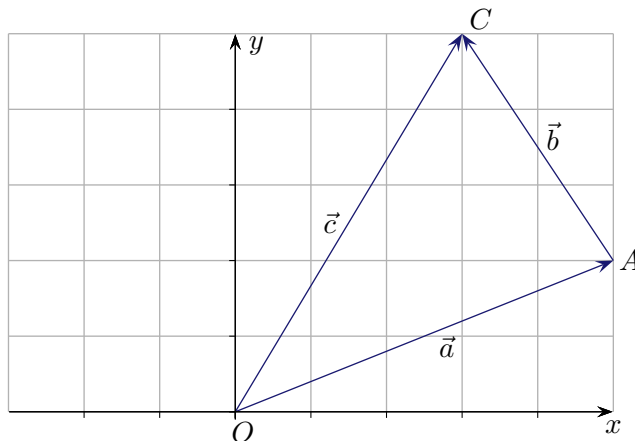


↑ Vectors and Points

Find the components of the vectors \vec{a} , \vec{b} and \vec{c}
and the coordinates of the points A , B and C .
All values are integers.



↑ Closed vector chain



It applies: $\vec{a} + \vec{b} = \vec{c}$
 $\vec{b} = \vec{c} - \vec{a}$
 $\vec{AC} = \vec{OC} - \vec{OA}$ difference vector

The direction vector \vec{AC} to points A and C is obtained with „tip minus tail“.

We get the same result with the following rule, which can be generalized.

Here we have a closed vector chain with $\vec{a} + \vec{b} - \vec{c} = \vec{0}$, the direction of the vectors is irrelevant.

Loop rule for a closed vector chain:

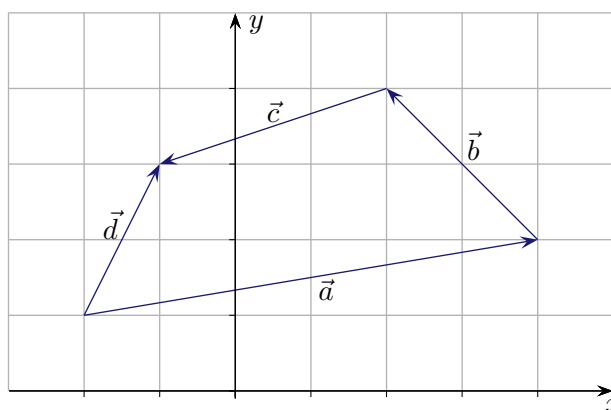
The arrow to \vec{b} starts in A and ends in C .

To go from A via O to C , the arrow to \vec{a} is in the reverse direction and the arrow to \vec{c} in the direction indicated to pass through: $\vec{b} = -\vec{a} + \vec{c}$ ($= \vec{c} - \vec{a}$)

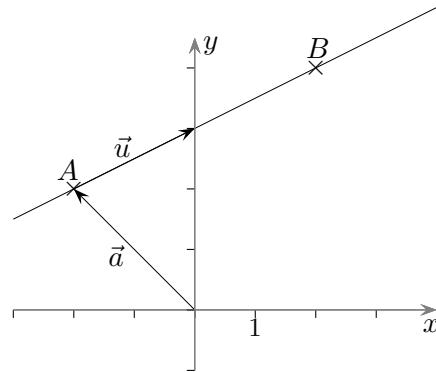
$$\vec{a} = \vec{d} - \vec{c} - \vec{b}$$

$$\vec{c} = -\vec{b} - \vec{a} + \vec{d}$$

$$\vec{d} = \vec{a} + \vec{b} + \vec{c}$$



↑ Equation of a line



The line g passes through the points $A(-2 | 2)$ and $B(2 | 4)$.

What is the vector line equation?

The line can be determined by the

support vector $\vec{a} = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$, which leads to the point A ,

and the direction vector $\vec{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ (or a multiple thereof) can be specified.

To each point of the line point the vectors:

$$\vec{a} + \vec{u} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$$\vec{a} + 2\vec{u} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$\vec{a} + 3\vec{u} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

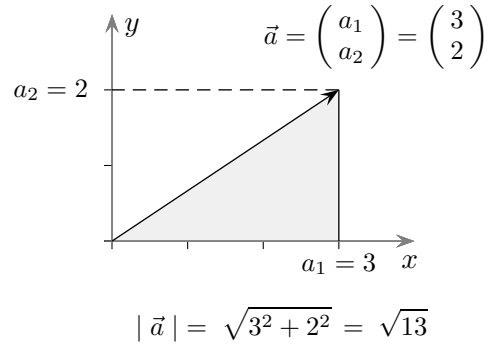
Die *line equation* is therefore: $\vec{x} = \vec{a} + k\vec{u}$ and for our example: $\vec{x} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} + k \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

For each k -value we get a vector \vec{x} which leads to a point $P(-2 + 2k | 2 + k)$ on the line.

↑ Magnitude (amount) of a vector

The length $|\vec{a}|$ or the magnitude of a vector $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ (2-dimensional) is: $|\vec{a}| = \sqrt{a_1^2 + a_2^2}$

and for $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ (3-dimensional) $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

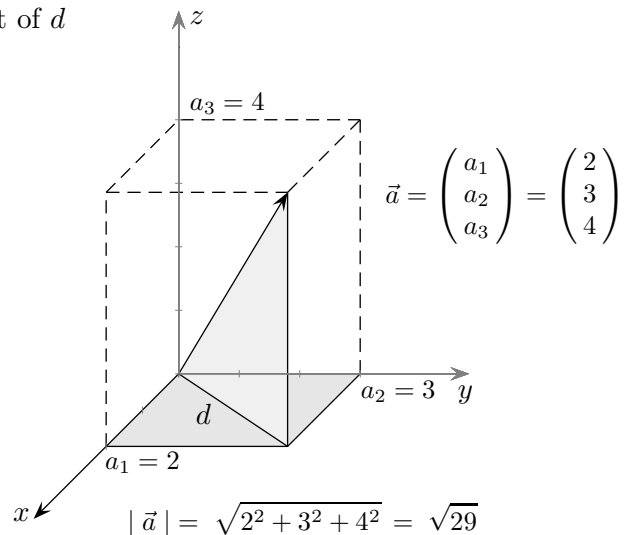


This is based on the calculation of the diagonal of a rectangle or the space diagonal of a box.

To calculate the length of the space diagonal of the box, an intermediate step is to determine the diagonal length $d = \sqrt{a_1^2 + a_2^2}$ of the ground area.

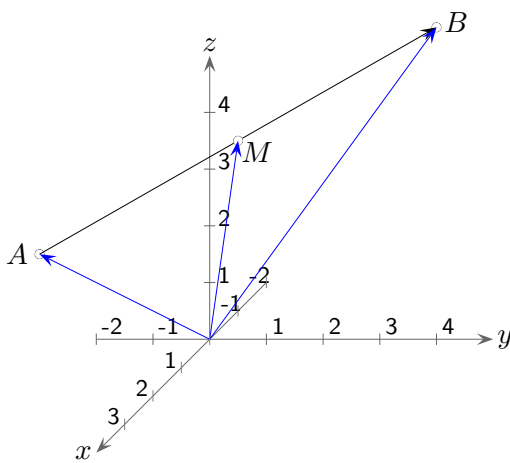
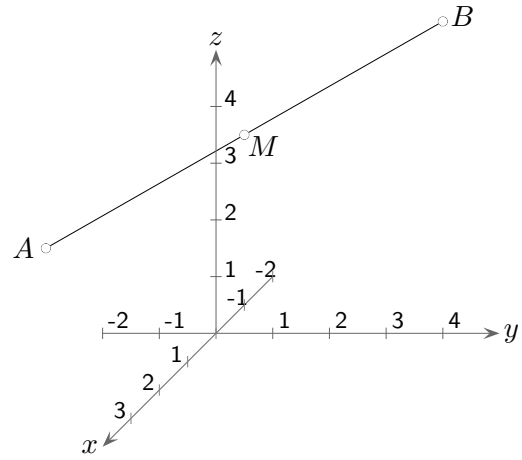
When applying the Pythagorean theorem again, the root of d falls away when squaring:

$$|\vec{a}| = \sqrt{d^2 + a_3^2} = \sqrt{a_1^2 + a_2^2 + a_3^2}$$



↑ Midpoint of a line segment

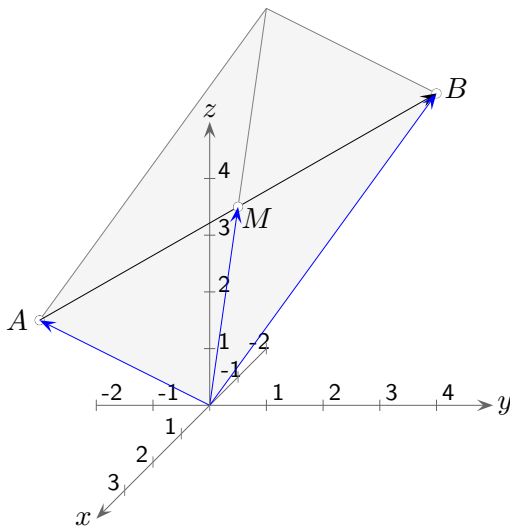
Determine the center of the line segment \overline{AB} with $A(4 \mid -1 \mid 3,5)$ and $B(-2 \mid 3 \mid 4,5)$.



$$\begin{aligned}\vec{OM} &= \vec{OA} + \frac{1}{2} \vec{AB}, & M(1 \mid 1 \mid 4) \\ \vec{M} &= \vec{A} + \frac{1}{2} \vec{AB}\end{aligned}$$

Point rule

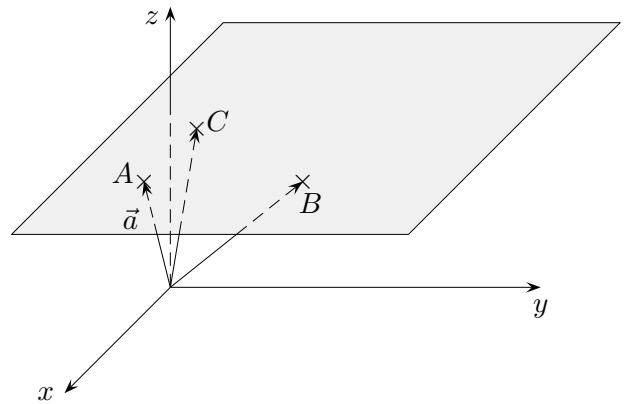
If the coordinates of a point are to be determined, the vector sum must contain at least one position vector as summand.



$$\begin{aligned}\vec{OM} &= \frac{1}{2} (\vec{OA} + \vec{OB}) \\ \vec{M} &= \frac{1}{2} (\vec{A} + \vec{B})\end{aligned}$$

↑ Plane equation

What is the equation of the plane in which the points $A(2 | 1 | -3)$, $B(1 | 5 | 0)$ and $C(4 | -1 | 2)$ are?



The direction of the plane can be determined by two direction vectors $\vec{u} = \overrightarrow{AB}$ and $\vec{v} = \overrightarrow{AC}$.

For the determination of the position only a support vector is necessary, for instance $\vec{a} = \overrightarrow{OA}$, \overrightarrow{OB} or \overrightarrow{OC} .

This gives us the *parameter form* of the plane equation:

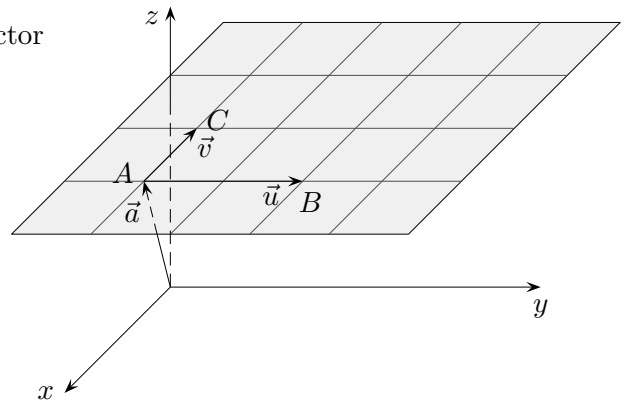
$$\vec{x} = \vec{a} + s\vec{u} + t\vec{v}$$

If s and t pass through all numbers, the vector \vec{x} reaches every point of the plane.

For the example:

$$\vec{u} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$$

$$\vec{v} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix}$$



The plane equation is:

$$\vec{x} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + s \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} + t \begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix}$$

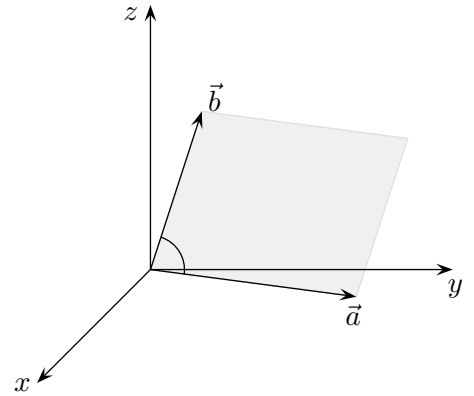
↑ Scalar product

There is a simple criterion to check if two vectors \vec{a} and \vec{b} are perpendicular to each other.

From the drawing below we can see that \vec{a} and \vec{b} are perpendicular to each other if the following holds:

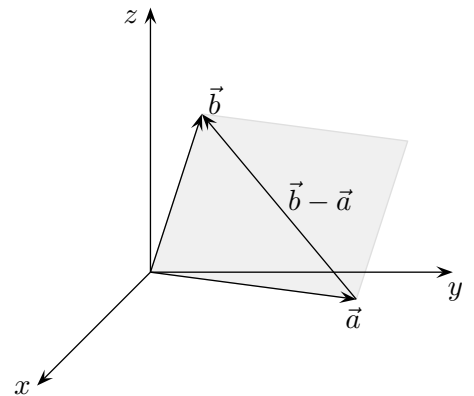
$$|\vec{a}|^2 + |\vec{b}|^2 = |\vec{b} - \vec{a}|^2 \text{ (Pythagorean theorem, the reverse).}$$

If we calculate the lengths using the components we get the condition we are looking for.



$$\text{Let } \vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \text{ and } \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\text{take note: } |\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$



$$\begin{aligned} \vec{a} \perp \vec{b} &\iff a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 \\ &= (b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2 \end{aligned}$$

\iff

- Expand brackets, in mind is sufficient, on both sides are the same squares
- summarize, simplify

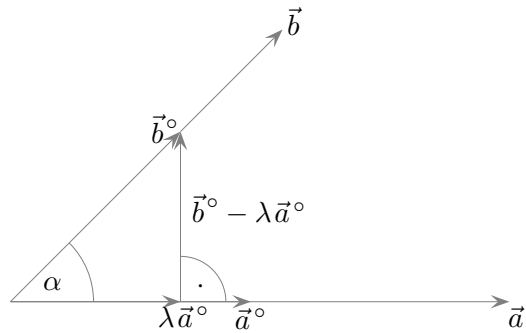
$$\iff a_1b_1 + a_2b_2 + a_3b_3 = 0$$

Definition of the scalar (dot) product:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1b_1 + a_2b_2 + a_3b_3$$

Theorem: $\vec{a} \perp \vec{b} \iff \vec{a} \cdot \vec{b} = 0$

↑ Relationship between vectors and the enclosed angle



$$\cos \alpha = \frac{|\lambda \vec{a}^\circ|}{|\vec{b}^\circ|} = \lambda$$

unit vektors $\vec{a}^\circ, \vec{b}^\circ$

$$\vec{b}^\circ - \lambda \vec{a}^\circ \perp \vec{a}^\circ$$

...

$$\lambda = \vec{a}^\circ \cdot \vec{b}^\circ$$

pay attention to: $\vec{a}^\circ \cdot \vec{a}^\circ = 1$

$$\cos \alpha = \vec{a}^\circ \cdot \vec{b}^\circ = \frac{\vec{a}}{|\vec{a}|} \cdot \frac{\vec{b}}{|\vec{b}|}$$

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos \alpha$$

Vectors of length 1 are called unit vectors. Each vector can be converted into the unit vector by dividing the vector by its length.

↑ Vector product

How do we find a vector \vec{n} perpendicular to the vectors

$$\vec{a} = \begin{pmatrix} 1 \\ -8 \\ 2 \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} 2 \\ -7 \\ 1 \end{pmatrix}?$$

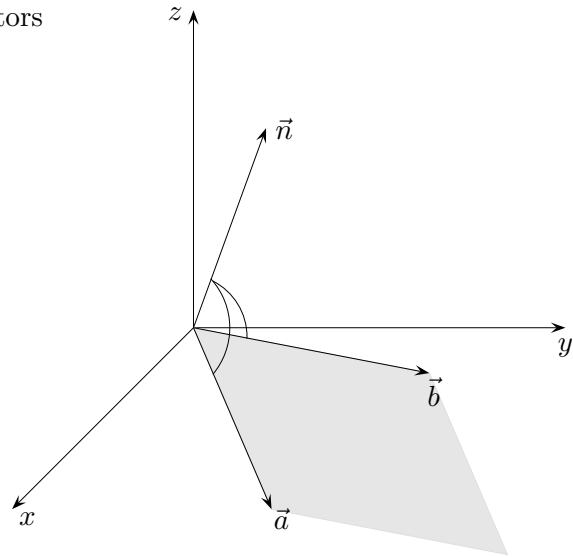
It must apply *I.* $\vec{a} \cdot \vec{n} = 0$

and *II.* $\vec{b} \cdot \vec{n} = 0$

with $\vec{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$ thus:

$$*I.* \quad n_1 - 8n_2 + 2n_3 = 0$$

$$*II.* \quad 2n_1 - 7n_2 + n_3 = 0$$



Since two equations with three variables are given, one variable can be freely chosen, for instance $n_1 = 2$

$$*I.* \quad -8n_2 + 2n_3 = -2$$

$$*II.* \quad -7n_2 + n_3 = -4$$

$$\begin{array}{l} \cdot \\ \cdot \\ \cdot \end{array} \quad n_2 = 1, \quad n_3 = 3, \quad \vec{n} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

So that we do not have to solve a system of equations again for each application, we perform the above calculation in general:

So let $\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ are given, we are searching for $\vec{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$

with $\vec{n} \perp \vec{a}$ and $\vec{n} \perp \vec{b}$,

that means $\vec{n} \cdot \vec{a} = 0$ and $\vec{n} \cdot \vec{b} = 0$.

↑ Vector product

$$\begin{aligned} I. \quad & a_1 n_1 + a_2 n_2 + a_3 n_3 = 0 \\ II. \quad & b_1 n_1 + b_2 n_2 + b_3 n_3 = 0 \end{aligned}$$

$$\begin{aligned} I. \quad & a_2 n_2 + a_3 n_3 = -a_1 n_1 \quad | \cdot (-b_2) \\ II. \quad & b_2 n_2 + b_3 n_3 = -b_1 n_1 \quad | \cdot a_2 \end{aligned}$$

$$\begin{aligned} I. \quad & -a_2 b_2 n_2 - a_3 b_2 n_3 = a_1 b_2 n_1 \\ II. \quad & a_2 b_2 n_2 + a_2 b_3 n_3 = -a_2 b_1 n_1 \end{aligned} \left. \vphantom{\begin{aligned} I. \\ II. \end{aligned}} \right\} +$$

$$n_3(a_2 b_3 - a_3 b_2) = (a_1 b_2 - a_2 b_1) n_1$$

$$n_3 = \frac{(a_1 b_2 - a_2 b_1) n_1}{a_2 b_3 - a_3 b_2}$$

$$n_2 = \frac{(a_3 b_1 - a_1 b_3) n_1}{a_2 b_3 - a_3 b_2}$$

To obtain the simplest possible result,
we choose $n_1 = a_2 b_3 - a_3 b_2$ and get

$$\vec{n} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

Definition of vector product (Cross product of vectors):

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

Examples:

$$\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \times \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \\ -8 \end{pmatrix}, \quad \begin{pmatrix} -3 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ -8 \\ -1 \end{pmatrix}$$

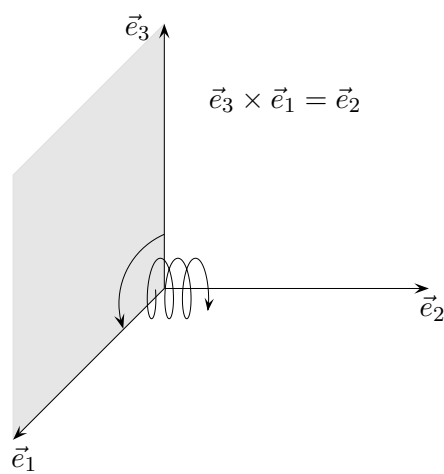
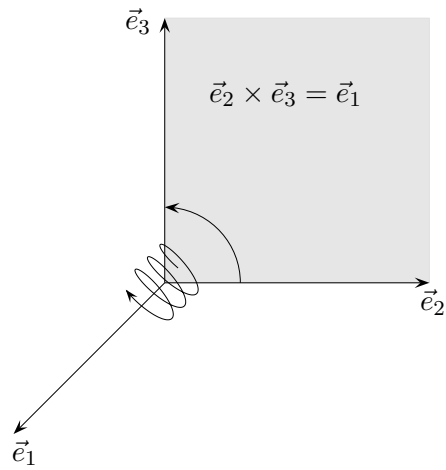
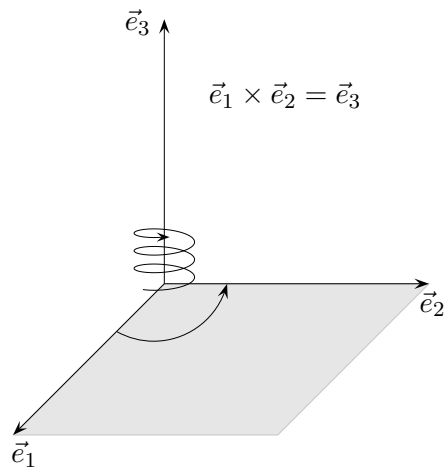
Memory rule:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ \dots \\ \dots \end{pmatrix}$$

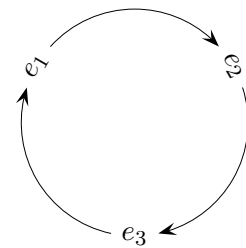
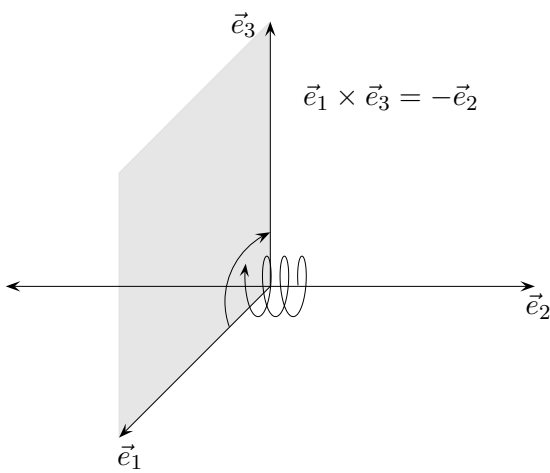
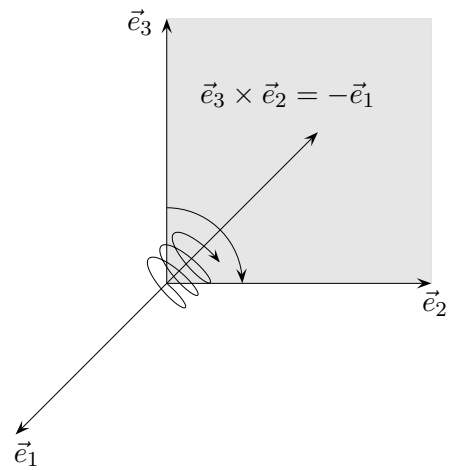
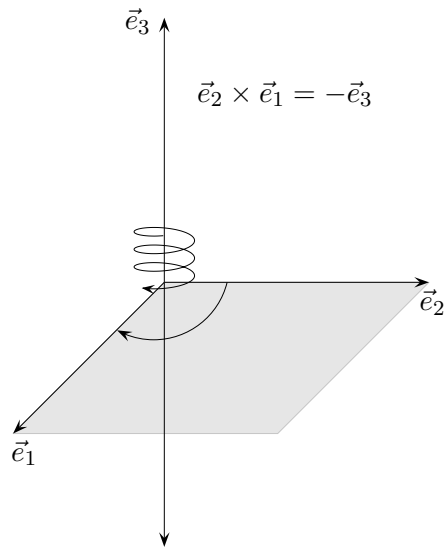
$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} \dots \\ a_3 b_1 - a_1 b_3 \\ \dots \end{pmatrix}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} \dots \\ \dots \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

↑ Orientation



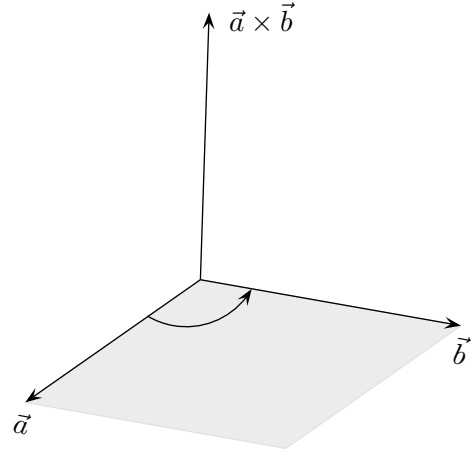
↑ Orientierung



↑ Vector product continued

Rules of calculation:

1. Exactly then $\vec{a} \times \vec{b} = \vec{0}$ if \vec{a}, \vec{b} are collinear.
2. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
4. $\vec{a} \times (\lambda \vec{b}) = \lambda(\vec{a} \times \vec{b})$
5. $\vec{a}, \vec{b}, \vec{a} \times \vec{b}$ form in this order, form a right system.



What is the significance of $|\vec{a} \times \vec{b}|$?

Examples like $\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ ab \end{pmatrix}$ suggest that the following applies:
 $|\vec{a} \times \vec{b}| = |\vec{a}| \cdot |\vec{b}|$ (Rectangle area), if $\vec{a} \perp \vec{b}$.

With a little arithmetic, the assumption can be confirmed:

$$\left| \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix} \right| = |\vec{a}| \cdot |\vec{b}| \iff (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 = (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$$

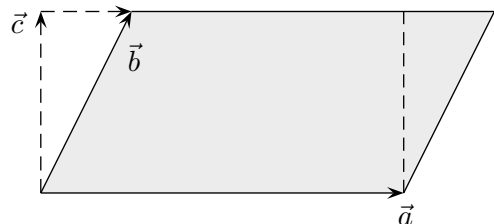
$$\iff -2a_2b_3a_3b_2 - 2a_3b_1a_1b_3 - 2a_1b_2a_2b_1 = a_1^2b_1^2 + a_2^2b_2^2 + a_3^2b_3^2$$

From assumption $\vec{a} \perp \vec{b}$ results $a_1b_1 + a_2b_2 + a_3b_3 = 0$, then applies too:

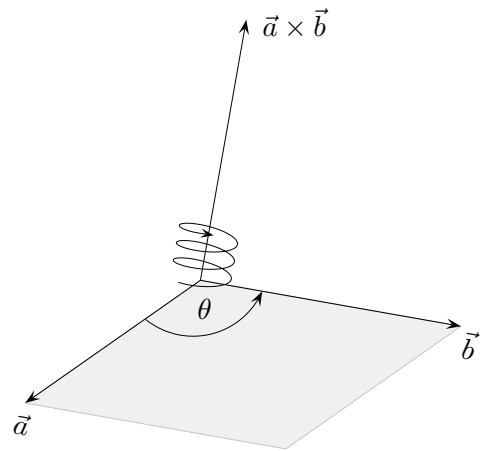
$$(a_1b_1 + a_2b_2 + a_3b_3)^2 = 0$$

$$\iff a_1^2b_1^2 + a_2^2b_2^2 + a_3^2b_3^2 + 2a_2b_3a_3b_2 + 2a_3b_1a_1b_3 + 2a_1b_2a_2b_1 = 0 \quad \text{All cleared?}$$

$$\begin{aligned} |\vec{a} \times \vec{b}| &= |\vec{a} \times (\vec{c} + \lambda \vec{a})| \\ &= |\vec{a} \times \vec{c} + \lambda \vec{a} \times \vec{a}| \\ &= |\vec{a}| \cdot |\vec{c}| \quad \text{Parallelogram area} \end{aligned}$$



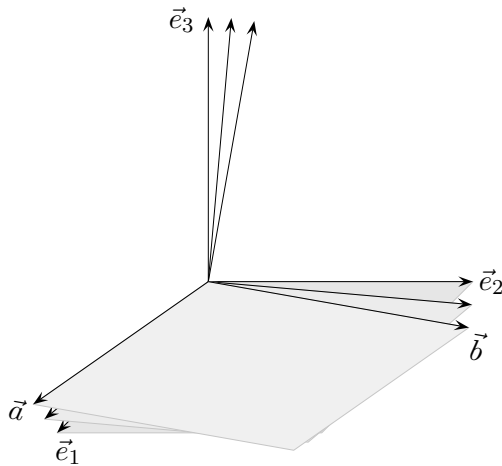
↑ Orientierung



Right-hand Rule

If the vector $\vec{a} \times \vec{b}$ is grasped in the right hand and the fingers curl around from \vec{a} to \vec{b} through the angle θ , the thumb points in the direction for $\vec{a} \times \vec{b}$.

To make it plausible:



\vec{e}_1, \vec{e}_2 can be continuously transformed into \vec{a}, \vec{b} so that the parallelogram area always remains positive. The orientation of the vector produkt then remains unchanged.

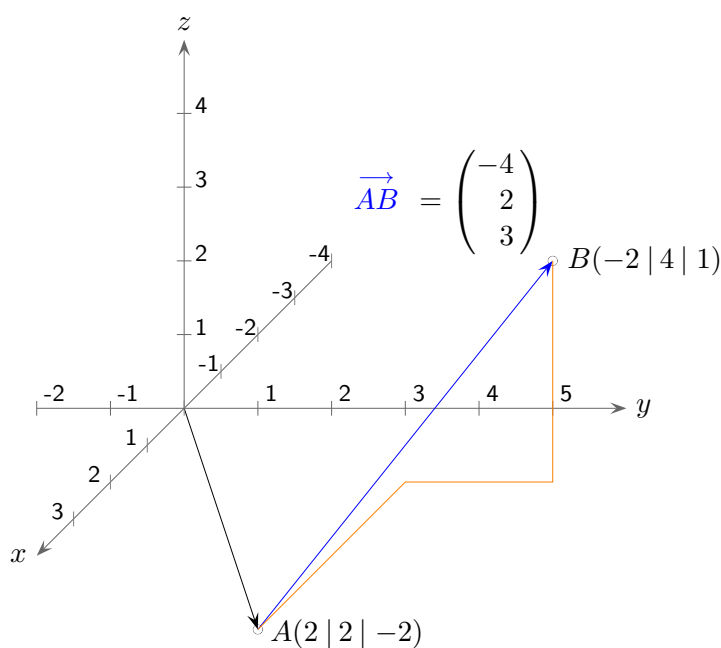
↑ Some maths

$\vec{\cdot}$ can be taken as a function that assigns to each ordered pair of points (A, B) (graphical an arrow), a vector of \mathbb{R}^3 :

$A(a_1 | a_2 | a_3), B(b_1 | b_2 | b_3)$

$\vec{AB} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \\ b_3 - a_3 \end{pmatrix}$ The vector contains the coordinate changes to get from A to B and the length of \overline{AB} .

The following applies for this assignment: $\vec{AB} + \vec{BC} = \vec{AC}$, $\vec{AB} = \vec{OB} - \vec{OA}$ and $\vec{OA} + \vec{AB} = \vec{OB}$



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