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↑ Real numbers $\mathbb{R}$

The diagonal length of a square with side length 1 is exactly $\sqrt{2}$ length unit.

$$\sqrt{2} = 1{,}4142135623730950488016887242096980785696718753769480731766797379907 32\ldots$$

We can consider $\sqrt{2}$ as a number with an infinite number of digits after the decimal point. This idea is facilitated if we remember that we can easily speak of the infinite set of natural numbers $\mathbb{N}$.

0,101001000100001000001 ...

$1/7 = 0{,}142857142857142857142857142857142857\ldots$ periodical

<u>All</u> 0/1/2/3 ... /9-sequences preceded by $a,\ldots$ ($a \in \mathbb{Z}$) form the set of real numbers.
Numbers are virtual, they only exist in our mind. Therefore, this construction is possible.
Real numbers that consist only of zeros from a position on (which can be omitted) are the finite decimal numbers. The real numbers are necessary because in theory there should be no limits of accuracy. But how are these infinite decimal numbers added and multiplied?

As an example we calculate π^2.

$$\begin{aligned}
\pi &= 3{,}14159265358979323846264338327950288419716939937510582097494459230781 6\ldots \\
\pi &\in [3{,}1415;\ 3{,}1416] \\
\pi^2 &\in [3{,}1415^2;\ 3{,}1416^2] \\
\pi^2 &\in [9{,}86902225;\ 9{,}86965056] \\
\pi^2 &= 9{,}869\ldots
\end{aligned}$$

next step

$$\begin{aligned}
\pi &\in [3{,}14159;\ 3{,}14160] \\
\pi^2 &\in [3{,}14159^2;\ 3{,}14160^2] \\
\pi^2 &\in [9{,}8695877281;\ 9{,}86965056] \\
\pi^2 &= 9{,}869\ldots \qquad \text{has been of no use}
\end{aligned}$$

next step

$$\begin{aligned}
\pi &\in [3{,}141592;\ 3{,}141593] \\
\pi^2 &\in [3{,}141592^2;\ 3{,}141593^2] \\
\pi^2 &\in [9{,}869600294464;\ 9{,}869606577649] \\
\pi^2 &= 9{,}86960\ldots
\end{aligned}$$

and so on.

We imagine that with this interval nesting to the decimal number approximations ($\in \mathbb{Q}$) of π successively the valid decimal digits of

$$\pi^2 = 9{,}86960440108935861883449099987615113531369940724079062641334937622004 2\ldots$$

emerge. In fact, we have sketched out the idea of how multiplication (analogous to addition) can be done, so that terms like 3π, $4\sqrt{5}$, $\sqrt{2}+\sqrt{7}$ are being explained.

$\uparrow$ Addition in $\mathbb{R}$

As an example we calculate $\sqrt{2} + \sqrt{7}$.

$$\begin{aligned} \sqrt{2} &= 1{,}4142135623730950488\ldots \\ \sqrt{7} &= 2{,}6457513110645905905\ldots \end{aligned}$$

$$\begin{aligned} \sqrt{2} &\in [1{,}4142;\ 1{,}4143] \\ \sqrt{7} &\in [2{,}6457;\ 2{,}6458] \\ \sqrt{2} + \sqrt{7} &\in [4{,}0599;\ 4{,}0601] \\ \sqrt{2} + \sqrt{7} &= 4{,}0\ldots \end{aligned}$$

next step

$$\begin{aligned} \sqrt{2} &\in [1{,}41421;\ 1{,}41422] \\ \sqrt{7} &\in [2{,}64575;\ 2{,}64576] \\ \sqrt{2} + \sqrt{7} &\in [4{,}05996;\ 4{,}05998] \\ \sqrt{2} + \sqrt{7} &= 4{,}0599\ldots \end{aligned}$$

next step

$$\begin{aligned} \sqrt{2} &\in [1{,}414213;\ 1{,}414214] \\ \sqrt{7} &\in [2{,}645751;\ 2{,}645752] \\ \sqrt{2} + \sqrt{7} &\in [4{,}059964;\ 4{,}059966] \\ \sqrt{2} + \sqrt{7} &= 4{,}05996\ldots \end{aligned}$$

and so on.

With this interval nesting emerge to the decimal number approximations of $\sqrt{2}$ und $\sqrt{7}$ successively the valid digits of $\sqrt{2} + \sqrt{7}$. To put it more elegantly:
The sum is the supremum (see page 4) of all left (monotonically increasing) interval boundaries, and the infimum of all right-hand interval boundaries.

$$\sqrt{2} = 1{,}41421356237309504880168 8$$

$$\sqrt{7} = 2{,}64575131106459059050161 57$$

$$\sqrt{2} + \sqrt{7} = 4{,}05996487343768563930330 44$$

The unbiased handling of real numbers as infinite decimal numbers facilitates access to limit considerations, which becomes necessary from the 11th grade onwards. The alternative, „elegant“ construction with Cauchy sequences or Dedekind cuts is reserved for the study of mathematics. This approach is more advanced, but by no means more „rigorous“. In an axiomatic introduction of the real numbers, for didactic reasons, the reference to infinite decimal numbers, as well as the simple calculation of the supremum (next page) for limited sets should not be missing.

Of a real number only an initial part is visible, but in principle of arbitrary length (even if this can require considerable effort).

Considering the table,
it is reasonable to assume that the (monotone) sequence $a_n = (1 + \frac{1}{n})^n$ converges[1].

As n increases, more and more valid decimal digits result[2].

n	$(1 + \frac{1}{n})^n$
10^1	2,593 742 46 ...
10^2	2,704 813 82 ...
10^3	2,716 923 93 ...
10^4	2,718 145 92 ...
10^5	2,718 268 23 ...
10^6	2,718 280 46 ...
10^7	2,718 281 69 ...
10^8	2,718 281 81 ...

Due to Euler(1707 - 1783) the limit $\lim\limits_{n\to\infty} a_n$ is called e (e from exponential).

e is of great importance in calculus because of $(e^x)' = e^x$.

The numbers in this sequence are rational.
The example proves the connection between limit values and real numbers.

$$e = 2{,}718281828459045235360\ldots$$

[1] The sequence is bounded. With the monotony follows the convergence (against the supremum, theorem of analysis).

[2] The proof which decimal digits are valid can be done with an interval nesting.

↑ Supremum/Infimum

The supremum of a set of numbers is the smallest upper bound.
The infimum of a set of numbers is the largest lower bound.

The following six statements are equivalent and characterise therefore equally the completeness of the real numbers:

1. Interval nesting principle
2. Every non-empty upper bounded subset of $\mathbb{R}$ has a supremum.
3. Every non-empty lower bounded subset of $\mathbb{R}$ has an infimum.
4. Every Cauchy sequence converges against a real number.
5. Each bounded sequence has an accumulation point (Bolzano-Weierstraß).
6. Every monotonically increasing upward bounded sequence converges.

Interval nesting principle
Let (I_n) be a sequence of closed, bounded intervals with the properties
(1) $I_1 \supseteq I_2 \supseteq \ldots,$ and
(2) the diameters of I_n tend towards 0.

Then there is exactly one real number a that lies in each interval I_n.

Let A be a non-empty, upper bounded subset of positive numbers from $\mathbb{R}$.

Proof of the supremum

The numbers from A are of the form $a{,}a_1a_2a_3a_4\ldots,\quad a \in \mathbb{N},\; a_i \in \{0,1,2,3,4,5,6,7,8,9\}$.
In A there exists a number with a maximal.
Among all numbers $a{,}a_1a_2a_3a_4\ldots$ from A there exists a number with a_1 maximal.
Among all numbers $a{,}a_1a_2a_3a_4\ldots$ from A there exists a number with a_2 maximal.
Among all numbers $a{,}a_1a_2a_3a_4\ldots$ from A there exists a number with a_3 maximal
and so on.

Visual:
The Supremum $a{,}a_1a_2a_3a_4\ldots$ is like a minimal envelope that is wrapped on top of A.

For a Cauchy sequence, the following is true:

$$\underset{\varepsilon>0}{\forall}\quad \underset{n_0\in\mathbb{N}}{\exists}\quad \underset{m,n\ge n_0}{\forall}\quad |\, a_m - a_n \,| < \varepsilon$$

From a certain position (n_0) the distance between the sequence elements is arbitrarily small.
From a certain position the sequence elements differ arbitrarily little of each other.

At 5.
Ongoing bisection of the interval (sequence limited!), whereby at least in one half there are infinitely many elements of the sequence, leads to an accumulation point.

↑ Limit

Examples:
$\sqrt{5}$, π, 0,101001000100001..., $\frac{1}{3} = 0,\overline{3}$

Since a real number has infinite number of digits after the decimal point, it can be difficult to grasp, if it is not a root or is not periodic or does not have a pattern. Way out: With a convergent sequence, a real number can be defined, it is then called the limit of the sequence.

$a = 0{,}101001000100001\dots$ is characterised by the sequence

$$\begin{aligned} a_1 &= 0{,}1 \\ a_2 &= 0{,}101 \\ a_3 &= 0{,}101001 \\ &\dots \end{aligned}$$

The sequence elements are approximations for a. The further one proceeds in the sequence, the better the approximations will be, and the more valid digits will emerge of the limit.

The sequence converges to a, because for every (arbitrarily small) neighbourhood (it is a measure of the deviation) of a there is a position in the sequence, from which all further sequence members lie in the neighbourhood.

Definition
The sequence $(a_n)_{n\in\mathbb{N}}$ converges (strives) against the limit a, written $\lim\limits_{n\to\infty} a_n = a$ or $a_n \to a$,
if holds true

$$\underset{\varepsilon>0}{\forall}\ \underset{n_0\in\mathbb{N}}{\exists}\ \underset{n\geq n_0}{\forall}\ |\, a_n - a\,| < \varepsilon \qquad \text{or in another notation:}$$

$$\forall\,\varepsilon > 0\ \ \exists n_0 \in \mathbb{N}\ \big(n \geq n_0 \implies |\, a_n - a\,| < \varepsilon\big)$$

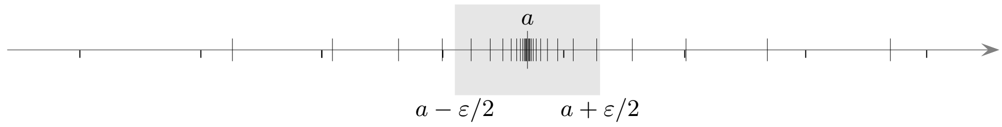
A convergent sequence defines a real number (limit) from which any number of digits can be calculated.
$|\, a_n - a\,| < \varepsilon$ means $a_n - \varepsilon < a < a_n + \varepsilon$.
For instance $\varepsilon = 10^{-5}$, from the related n_0 onwards (at least) the first 4 digits after the comma of a_n and a coincide,

$$\_,a_n^1a_n^2a_n^3a_n^4(a_n^5-1)\dots < \_,a^1a^2a^3a^4a^5\dots < \_,a_n^1a_n^2a_n^3a_n^4(a_n^5+1)\dots$$

provided that the 5th a_n-digits after the comma a_n^5 is not 0 or 9. If necessary, n is to be chosen larger.
Otherwise, for instance $a = 1{,}000001$, $a_n = 0{,}9999$ and $|\, a_n - a\,| < 10^{-5}$ is possible.

A sequence with the convergence behavior of the definition and computable n_0
is an algorithm to compute a real number to an arbitrarily specified number of digits.
However, because of the ambiguity of the number representation, it must be more precise:
... is an algorithm, to obtain an approximation for a real number with arbitrarily given (small) difference.

↑ Cauchy sequence



Every convergent sequence is a Cauchy sequence.

According to the condition, from a certain position onwards all sequence members lie in the $\varepsilon/2$-environment of a. For these sequence elements the distance from each other must then be smaller than ε.

The reversal is more important.
Each Cauchy sequence has a limit.

A Cauchy sequence is bounded and therefore contains a convergent subsequence.

$|\, a_m - a_n \,| < \varepsilon$ means $a_n - \varepsilon < a_m < a_n + \varepsilon$.
That is, for $\varepsilon = 10^{-k}$, starting from the related n_0, (at least) the first $k-1$ digits after the decimal point of a_n and a_m agree with each other. (for all $m > n$, provided the kth a_n-digit after the decimal point is not 0 or 9).
This ensures that the remaining sequence members are arbitrarily close to the subsequence.

Plausible: A Cauchy sequence thus defines a number (a limit) a.

The formal proof can be found in many calculus scripts.

illustrative:
Cauchy sequences are sequences whose fluctuations become arbitrarily small,
which at some position begin to “tread water“.

Epsilontic initially is a hurdle that unfortunately often has to be taken without being prepared.
Let us assume that the equality of two real numbers a and b is to be proved. This is not easy to do due to the infinitely many decimal digits. But if it were possible, for every $\varepsilon > 0$ to verify the inequality $|\, b - a \,| < \varepsilon$, $a = b$ would have to hold.
Let us further assume that the number b is defined by a sequence a_n. Then for every $\varepsilon > 0$ there would have to be an index n_0, so that for all further sequence elements $|\, a_n - a \,| < \varepsilon$ holds.
Now the jump to the definition (Weierstrass 1815-1897) is not far:

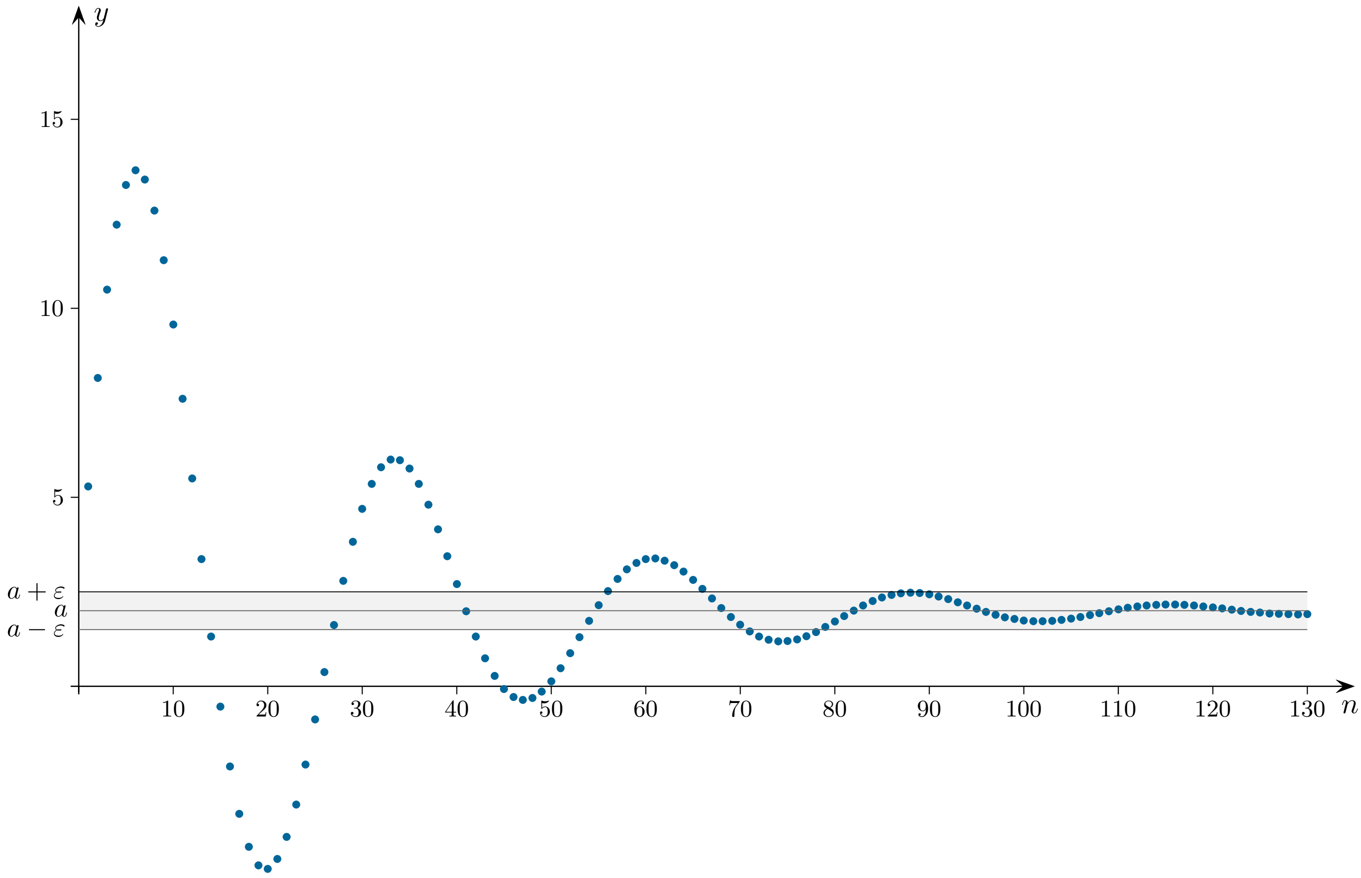
The sequence $(a_n)_{n\in\mathbb{N}}$ converges against the limit a, written $\lim\limits_{n\to\infty} a_n = a$, if applies:

$$\underset{\varepsilon>0}{\forall} \quad \underset{n_0\in\mathbb{N}}{\exists} \quad \underset{n\geq n_0}{\forall} \quad |\, a_n - a \,| < \varepsilon$$

The sequence then defines the value a.

In the ε-n_0 proof n_0 is to be represented as a function of ε, the notation $n_0(\varepsilon)$ makes this clear. It is not necessary to specify this function explicitly or to search for the smallest possible n_0. It is sufficient to show that for every $\varepsilon > 0$ *there is* such a n_0. The ε-n_0 proof does not depend on large ε. If the ε-n_0 proof applies is valid for all $0 < \varepsilon < \varepsilon_0$ with any $\varepsilon_0 > 0$, it is also valid for all $\varepsilon > 0$. For convergence questions, the first sequence members are irrelevant. Finally, in the ε-n_0 proof, it does not matter whether is used $|a_n - a| < \varepsilon$ or $|a_n - a| \leq \varepsilon$, $n \geq n_0$ or $n > n_0$. All possible formulations are equivalent.

↑ Visualisation



Such a graph only partially visualises the definition of the limit value $\lim\limits_{n\to\infty} a_n = a$.

The convergence condition states that for every (no matter how small) $\varepsilon > 0$ there exists an index n_0, so that from the position n_0 onwards (means for $n \geq n_0$) all a_n lie in the interval $[a-\varepsilon, a+\varepsilon]$, the points (n, a_n) then lie in the ε strip.

For any $\varepsilon > 0$, the points (n, a_n) ultimately lie in the ε-strip.

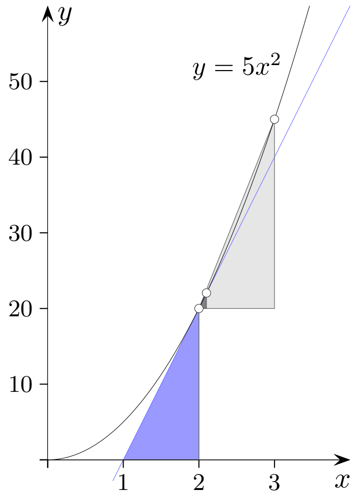
Equivalent:
For each $\varepsilon > 0$, only finitely many points (n, a_n) lie outside the ε strip.

The graph only gives the relation for <u>one</u> ε. By the condition: for <u>each</u> $\varepsilon > 0$ ... a real number a is defined by the sequence. Unlimited further valid decimal digits of (here) $2 = 1{,}9999\ldots = 2{,}0000\ldots$ are created.

$$
\begin{aligned}
a_{100} &= 1{,}742959 \\
a_{200} &= 2{,}005546 \\
a_{300} &= 1{,}999985 \\
a_{400} &= 1{,}999998 \\
a_{425} &= 1{,}9999996 \\
a_{450} &= 2{,}00000006 \\
&\ldots
\end{aligned}
$$

↑ Calculus

We imagine dropping a stone from a high rise building and are interested in the relationship between the passed time x (in seconds) and the distance of fall y (in metres).
The graph was created on the basis of the measured values.



What is the velocity of the stone at time (for instance) $x = 2$?
It seems obvious that at any given moment there is a velocity, but on the other hand we understand the (average) velocity Δv to be the quotient of the distance moved and the time passed. For a calculation therefore, in addition to $x = 2$ an further time is required.
The problem is that the speed determined in this way depends on the choice of the second time point[1].
By considering the sequence of approximations Δv, we achieve the velocity at the time $x = 2$.

x	2	3	2,1	2,01	2,001	2,0001	2,00001	$\ldots$
$y = 5x^2$	20	45	22,05	20,2005	20,020005	20,00200005	20,0002000005	
$\Delta v = \frac{y-20}{x-2}$		25	20,5	20,05	20,005	20,0005	20,00005	

This procedure is typical for calculus.
Ultimately, (instantaneous) velocity, area, etc. are <u>defined</u> in this way.

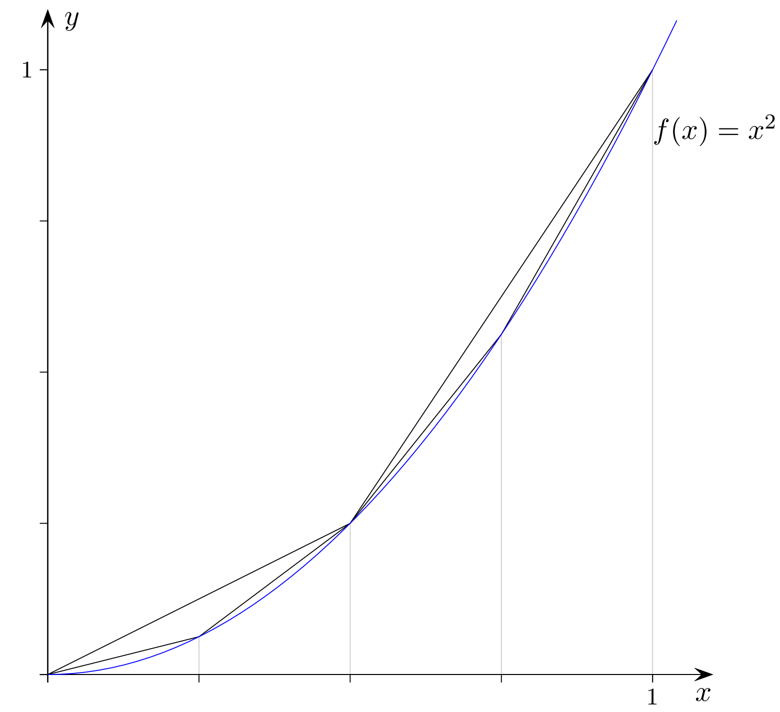
$$20 = 20{,}0000000000000000\ldots$$

[1] Even if the 2. time is very close to 2, a blue coloured gradient triangle remains visible.

↑ As another example, for the length of the arc of $f(x) = x^2$ on the interval $[0;1]$ an approximation sequence is determined, from which the first seven digits after the decimal point of the limit are obtained.

$$
\begin{aligned}
b_2 &= 1{,}46040481\ldots\\
b_4 &= 1{,}47428047\ldots\\
b_8 &= 1{,}47777798\ldots\\
b_{16} &= 1{,}47865168\ldots\\
b_{32} &= 1{,}47887006\ldots\\
b_{64} &= 1{,}47892466\ldots\\
b_{128} &= 1{,}47893830\ldots\\
b_{256} &= 1{,}47894172\ldots\\
b_{512} &= 1{,}47894257\ldots\\
b_{1024} &= 1{,}47894278\ldots\\
b_{2048} &= 1{,}47894283\ldots\\
&\ldots\\
\longrightarrow &\quad 1{,}47894285\ldots
\end{aligned}
$$

b_n is the length of the stretch line for n subdivisions.



Without justification, let us mention that this bounded, monotonically increasing sequence has the limit $\dfrac{\sqrt{5}}{2} + \dfrac{\operatorname{arcsinh}(2)}{4}$.

A sequence $a_1, a_2, a_3, a_4, \ldots$
can be transformed into an infinite sum with unchanged limit behaviour:

$$\underbrace{\underbrace{\underbrace{a_1 + (a_2 - a_1)}_{a_2} + (a_3 - a_2)}_{a_3} + (a_4 - a_3)}_{a_4} + \ldots$$

An infinite sum thus includes the sequence (a_n) of the sums of the first n summands (partial sums).

$$e^x = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ \ldots$$

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - \ \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + - \ \ldots$$

In this preferred manner, certain real numbers are grasped.

For all practical calculations, only computable real numbers are used (any number of digits can be determined). They form their own number range (countable field). It requires more effort to specify non-computable real numbers. The knowledge about this, as well as considerations about the magnitude, are of theoretical nature and irrelevant for applications. The existence of a supremum for bounded sets, which is necessary for a complete structure of calculus, has been shown.

↑ Limit of a geometric series

$s = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} \pm \ \ldots$

$$
\begin{aligned}
s_1 &= 1 \\
s_2 &= 0{,}5 \\
s_3 &= 0{,}75 \\
s_4 &= 0{,}625 \\
s_5 &= 0{,}6875 \\
s_6 &= 0{,}65625 \\
s_7 &= 0{,}671875 \\
s_8 &= 0{,}6640625 \\
s_9 &= 0{,}66796875 \\
s_{10} &= 0{,}66601562\ldots \\
s_{11} &= 0{,}66699219\ldots \\
s_{12} &= 0{,}66650391\ldots \\
s_{13} &= 0{,}66674805\ldots \\
s_{14} &= 0{,}66662598\ldots \\
s_{15} &= 0{,}66668701\ldots \\
s_{16} &= 0{,}66665649\ldots \\
s_{17} &= 0{,}66667175\ldots \\
s_{18} &= 0{,}66666412\ldots \\
s_{19} &= 0{,}66666794\ldots \\
s_{20} &= 0{,}66666603\ldots \\
s_{21} &= 0{,}66666698\ldots \\
s_{22} &= 0{,}66666651\ldots \\
s_{23} &= 0{,}66666675\ldots \\
s_{24} &= 0{,}66666663\ldots \\
& \ldots \\
\longrightarrow & \quad 0{,}66666666\ldots = \frac{2}{3}
\end{aligned}
$$

s_n is the sum of the first n summands.

The convergent series (see interval nesting) defines a real number s, the limit.
s is approximated arbitrarily exactly by s_n. The approximations s_n produce the real number s.
A convergent series (sequence) and its limit are to be considered as a unity.

↑ $0{,}999999\ldots = 1$

$$0{,}99999999999999999999\ldots$$

Numbers whose digits finally consist only of the 9 have a second representation.

$$\begin{aligned} x &= 0{,}999\ldots \\ 10x &= 9 + 0{,}999\ldots \\ 9x &= 9 \\ x &= 1 \end{aligned}$$

$$\begin{aligned} \frac{1}{9} &= 0{,}1111\ldots \quad | \cdot 9 \\ 1 &= 0{,}9999\ldots \end{aligned}$$

The proof can be even more elementary.

$0{,}\overline{9}$ is generated by the sequence

$$\begin{aligned} a_1 &= 0{,}9 \\ a_2 &= 0{,}99 \\ a_3 &= 0{,}999 \\ a_4 &= 0{,}9999 \\ a_5 &= 0{,}99999 \\ a_6 &= 0{,}999999 \\ &\ldots \end{aligned}$$

and

$1 - 0{,}\overline{9}$ by

$$\begin{aligned} d_1 &= 0{,}1 \\ d_2 &= 0{,}01 \\ d_3 &= 0{,}001 \\ d_4 &= 0{,}0001 \\ d_5 &= 0{,}00001 \\ d_6 &= 0{,}000001 \\ &\ldots \\ \longrightarrow &\quad 0{,}000000\ldots = 0 \end{aligned}$$

↑ Infinite series Telescope sum

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \ldots = 1$$

$$\begin{aligned}
\frac{1}{k(k+1)} &= \frac{1}{k} - \frac{1}{k+1} \\
s_n &= \sum_{k=1}^{n} \frac{1}{k(k+1)} = \Big(1 - \frac{1}{2}\Big) + \Big(\frac{1}{2} - \frac{1}{3}\Big) + \Big(\frac{1}{3} - \frac{1}{4}\Big) + \ldots + \Big(\frac{1}{n} - \frac{1}{n+1}\Big) \\
&= 1 - \frac{1}{n+1}
\end{aligned}$$

The sum term for s_n is considerably simplified by parentheses. The limit of the infinite series is reduced to the limit of the number sequence s_n:

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} s_n = 1$$

$$\begin{aligned}
e_n = \sum_{k=0}^{n} \frac{1}{k!} &= 1 + 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \ldots + \frac{1}{n!} \\
&\le 1 + 1 + \big(\tfrac{1}{2}\big)^1 + \big(\tfrac{1}{2}\big)^2 + \big(\tfrac{1}{2}\big)^3 + + \ldots + \big(\tfrac{1}{2}\big)^{n-1}, && \frac{1}{\ell} \le \frac{1}{2} \quad \text{für } \ell \ge 2 \\
&= 1 + \frac{1 - \big(\tfrac{1}{2}\big)^n}{1 - \tfrac{1}{2}} = 1 + 2 - \big(\tfrac{1}{2}\big)^{n-1} < 3 && \text{mit } q = \frac{1}{2}
\end{aligned}$$

Thus the sequence e_n is bounded upwards and obviously monotonically increasing, $\lim\limits_{n\to\infty} = e$.

Improved estimation

$$\begin{aligned}
e_n &\le 1 + 1 + \frac{1}{2} + \frac{1}{2}\cdot\big(\tfrac{1}{3}\big)^1 + \frac{1}{2}\cdot\big(\tfrac{1}{3}\big)^2 + \frac{1}{2}\cdot\big(\tfrac{1}{3}\big)^3 + \ldots + \frac{1}{2}\cdot\big(\tfrac{1}{3}\big)^{n-2}, && \frac{1}{\ell} \le \frac{1}{3} \quad \text{für } \ell \ge 3 \\
&= 1 + 1 + \frac{1}{2}\cdot\frac{1 - \big(\tfrac{1}{3}\big)^{n-1}}{1 - \tfrac{1}{3}} = 1 + 1 + \frac{3}{4} - \frac{1}{4}\cdot\big(\tfrac{1}{3}\big)^{n-2} < 2{,}75 && \text{mit } q = \frac{1}{3}
\end{aligned}$$

↑ Historical

The decadic number system is of Indian origin and was adopted by Arab scholars around 800.
The Flemish mathematician and engineer Simon Stevin 1548-1620 showed the advantages of calculating with decimals so that their use finally became established in the 16th century. John Napier used the decimal point notation in 1617.
René Descartes 1596-1650 and Pierre de Fermat 1607-1665 introduced the coordinate system around 1637, thus combining geometry and algebra for the first time. On the (continuous) number line, by defining a unit distance (unit of coordinates) a number is assigned to each point in a reversible and unambiguous way. The real numbers are called the *continuum*.
In contrast to Dedekind and Cauchy, Weierstrass 1815-1897 used infinite decimal representations in his lectures for the construction of $\mathbb{R}$.[1] There are several novel realisations of this idea. Addition and multiplication have to be defined, the rules and completeness must be proved, see Blatter or Singh.

[1] Decimal numbers $d_0, d_1d_2d_3\ldots$ were interpreted as (sometimes finite) set $\{\frac{d_0}{10^0}, \frac{d_1}{10^1}, \frac{d_2}{10^2}, \ldots\}$.
This definition was further extended (numerator and denominator could be any natural numbers).
As transcripts indicate, his students were probably unable to follow him in his further presentations.

Reelle Zahlen 11. Jg.
Grenzwert, siehe auch letzte Seite: Zusammengefasst
Startseite