# Leibniz-Regel für Parameter-Integrale

a) Leibniz-Regel

$$\frac{d}{dt}\int_a^b f(x,t)\,dx = \int_a^b \frac{\partial f(x,t)}{\partial t}\,dx$$

$$\frac{\displaystyle\int_a^b f(x,t+dt)\,dx - \int_a^b f(x,t)\,dx}{dt} = \frac{dt\displaystyle\int_a^b \tfrac{\partial f(x,t)}{\partial t}\,dx}{dt}$$

Für $f(x,t+dt)$ wird die tangentiale Approximation $f(x,t+dt) = f(x,t) + \dfrac{\partial f(x,t)}{\partial t}\,dt$ eingesetzt.

b) Leibniz-Regel, allgemein

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\,dx = f(b(t),t)\,b'(t) - f(a(t),t)\,a'(t) + \int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t}\,dx$$

Die Herleitung erfolgt mit dem Hauptsatz $\dfrac{d}{dx}\displaystyle\int_a^x g(t)\,dt = g(x)$ und der allgemeinen Kettenregel:

$$\begin{aligned}
\frac{d}{dt}\,G(c(t),d(t)) &= G_u(c(t),d(t))\cdot c'(t) + G_v(c(t),d(t))\cdot d'(t) \qquad \text{mit} \quad G(u,v)\\
G(u,t) &= \int_0^u f(x,t)\,dx\\
G_u(u,t) &= f(u,t)\\
G_v(u,t) &= \int_0^u \frac{\partial f(x,t)}{\partial t}\,dx \qquad\qquad \text{nach a)}\\
\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\,dx &= \frac{d}{dt}\big[G(b(t),t) - G(a(t),t)\big]\\
&= G_u(b(t),t)\cdot b'(t) + G_v(b(t),t)\cdot 1 - \big(G_u(a(t),t)\cdot a'(t) + G_v(a(t),t)\cdot 1\big)\\
&= f(b(t),t)\cdot b'(t) - f(a(t),t)\cdot a'(t) + \underbrace{\int_0^{b(t)} \frac{\partial f(x,t)}{\partial t}\,dx - \int_0^{a(t)} \frac{\partial f(x,t)}{\partial t}\,dx}_{\displaystyle\int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t}\,dx}
\end{aligned}$$

# Beispiel

Das Ergebnis von

$$\frac{d}{dt}\int\limits_0^{e^t}(t + x\sin x)\,dx = ?$$

lässt sich mit $a'(t) = 0$ und $\dfrac{\partial f(x,t)}{\partial t} = 1$ unmittelbar angeben:

$$f(b(t),t)\,b'(t) - f(a(t),t)\,a'(t) + \int\limits_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t}\,dx = (t + e^x \sin e^x)\,e^x + e^x$$